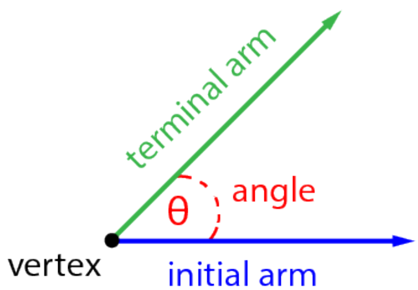
Grade 6 Math Circles
March 28/29/30, 2023
All About Angles

Taking an Angle on Angles

Objects around us come in all different shapes and sizes. Some have curves, some have edges; those with edges have corners and straight sides.

An **angle** is the space between two intersecting lines. These lines are called **arms** (or **rays**), and the point that they meet at is called the **vertex**. Conventionally, one of the arms is a horizontal line known as the **initial arm**, and the other one is called the **terminal arm**. In this lesson, any angles we will look at are going to be *positive* angles; that is, the angles formed by rotating the terminal arm *counterclockwise*.

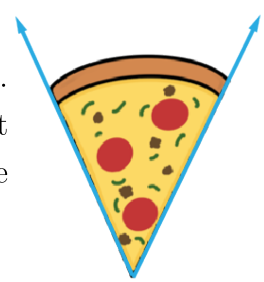


Stop and Think

What kinds of angles do you know of?

We measure many things. We measure time using seconds, minutes, hours, etc. We measure the length of objects in centimeters or meters. We can measure angles as well, and while there are different ways of doing this, today we will be talking about measuring angles with *degrees*.

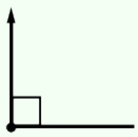
In a previous session about Transformations, we talked about rotations. A full rotation is 360°, which is essentially a circle! So we can look at angles as representing a slice of the circle (just as you can have a slice of pizza!).

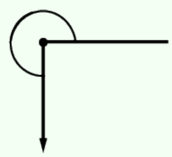
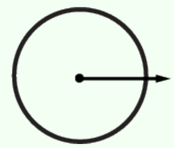


Example A

Draw the following angles:

1. 90°
2. 180°
3. 270°
4. 360°



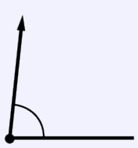
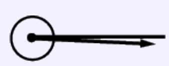



Are these the only angles that exist? Certainly not! In fact, there is an *infinite* amount of angles. This can seem daunting, which is why we categorize them according to certain properties. We use angles 1, 2 and 4 from Example A as "benchmarks" for characterizing various angles:

Acute angle: An angle that has a measure less than 90°.
Right angle: An angle that has a measure of exactly 90°.
Obtuse angle: An angle that has a measure greater than 90° but less than 180°.
Straight angle: An angle that has a measure of exactly 180°
Reflex angle: An angle that has a meaure greater than 180° but less than 360°.
Complete angle (Full Rotation): An angle that has a measure of exactly 360°.

Exercise 1

Classify the following angles:

1. 
2. 
3. 
4. 

Exercise 1 Solution

1. Acute angle
2. Reflex angle
3. Reflex angle
4. Straight angle

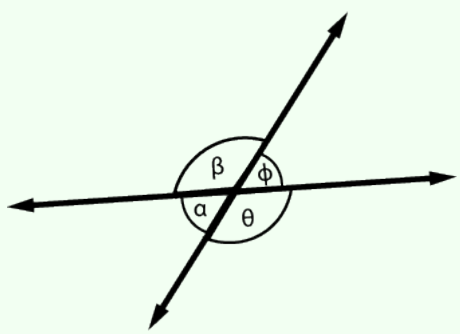
Complementary and Supplementary Angles

As fun as it can be to look at angles by themselves and characterize them, what we are actually interested in is looking at the *relationships* between angles. Below are a few definitions that will help us as we investigate them:

Adjacent angles: Two angles that are seperated by one line and share a vertex.
Complementary angles: Two adjacent angles that add up to $90°$
Supplementary angles: Two adjacent angles that add up to $180°$

Example B

Opposite angles are the angles that appear opposite each other when two lines intersect. In the diagram below, β and θ are opposite, and α and ϕ are opposite. Using the definitions we talked about above, what property can we say about opposite angles?



Looking more closely, we can see that the two intersecting lines creating four pairs of supplementary angles: $(\alpha, \beta), (\alpha, \theta), (\phi, \beta)$ and (ϕ, θ). From our definition of supplementary angles, this means that $\alpha + \beta = 180°$ and $\phi + \beta = 180°$, so it must be that $\alpha = \phi$.

Similarly, $\alpha + \beta = 180°$ and $\alpha + \theta = 180°$ which means that $\beta = \theta$.

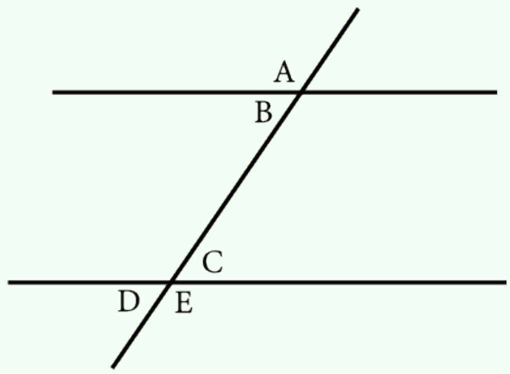
From this we can conclude that opposite angles must always be equal.

But wait, there's more! Recall that two lines **intersect** if they cross at exactly one point, and two lines are **parallel** if they can never intersect (represented with arrow ticks). What happens when we create angles by drawing one line that intersects two other parallel lines?

Example C

There are three angle relationships that we can point out. Using a protractor, measure the following angles and fill in the table. What do you notice?

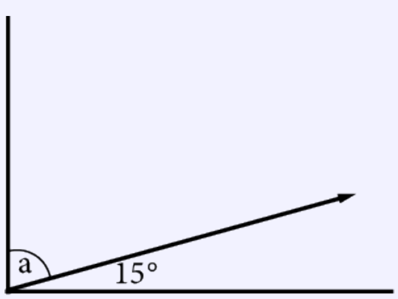
Angle	Measure
A	120°
B	60°
C	60°
D	60°
E	120°

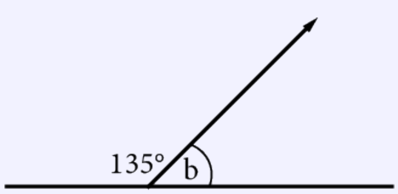


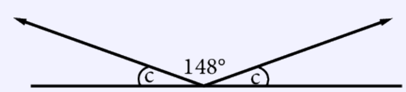
A way that we call this kind of diagram is *Z pattern* (since the parallel lines and the intersecting like form a ‘Z’ shape!). We give special names to the angles in a Z pattern. Angles B and D are **corresponding angles**; angles B and C are **interior alternate angles**; angles A and E are **exterior alternate angles**.

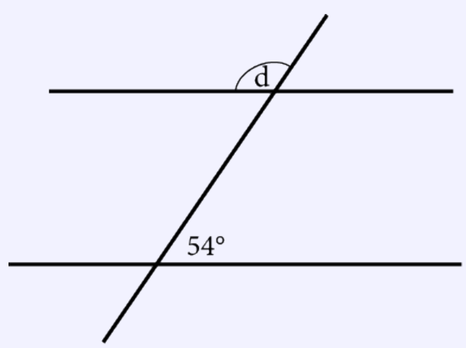
Exercise 2

Determine the measure of the unknown angles, then state which property you used:

1. 

2. 

3. 

4. 

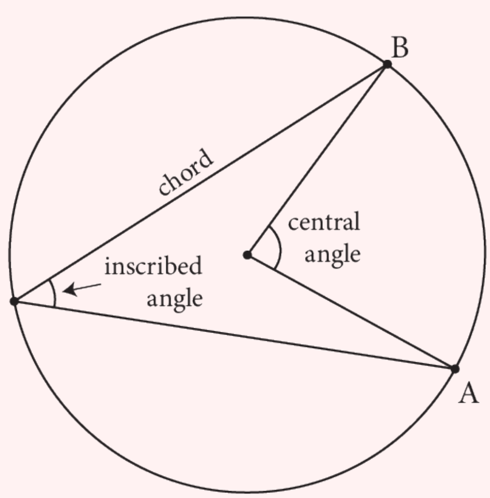
Exercise 2 Solution

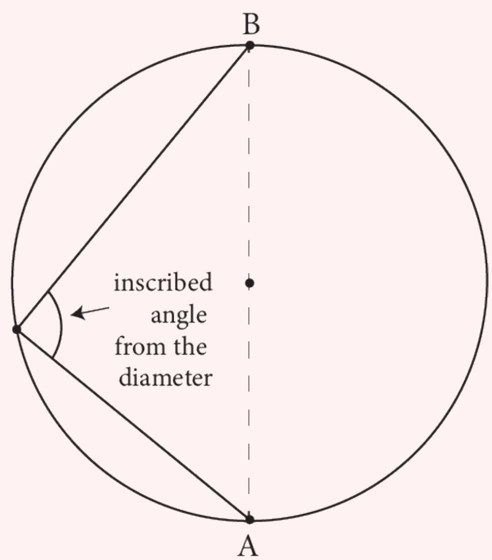
1. a = 75° due to the complementary angle property
2. b = 45° due to the supplementary angle property
3. a = 16° due to the straight angle and supplementary angle properties
4. a = 126° due to corresponding and supplementary angle properties

Angles in Circles

As discussed at the beginning of this lesson, a full rotation has an angle of 360° and creates a circle. Besides looking at the angles made from "cutting up" a circle, we can look at angles that are *inside* of circles.

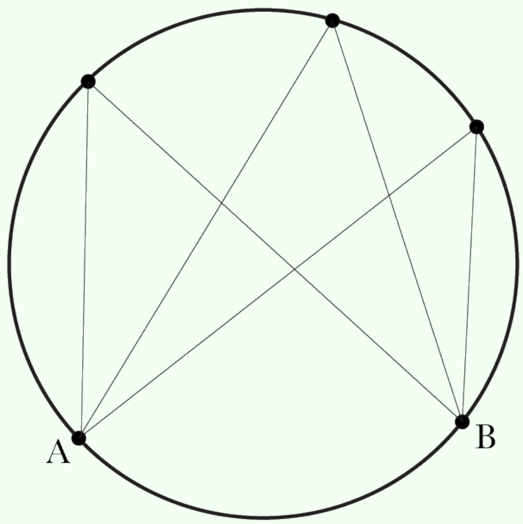
Chord: A line in the interior of a circle whose endpoints lie on the circumference of the circle.
Inscribed angle: An angle whose vertex lies on the circumference of the circle, formed by two intersecting chords.
Inscribed angle from the diameter: An inscribed angle formed by chords that each have one endpoint on the diameter of the circle.
Central angle: An angle whose vertex lies in the center of the circle, formed by two intersecting lines.





Example D

Draw three different inscribed angles from the same points A and B:

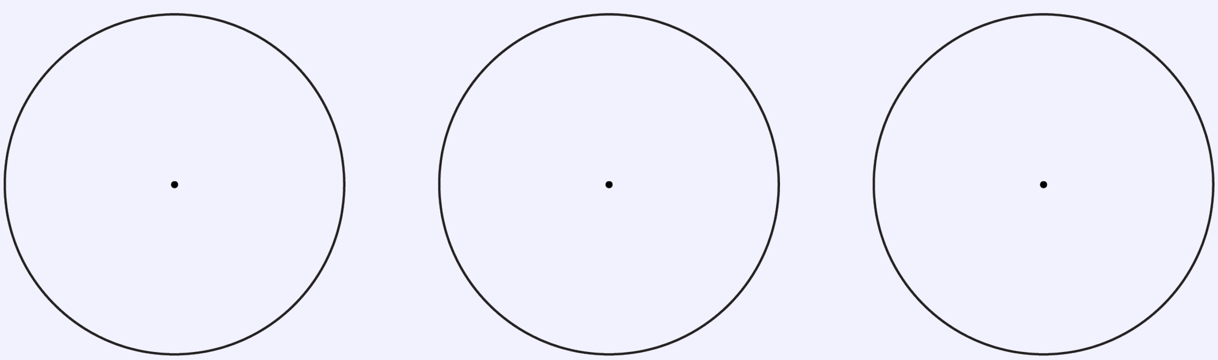


Stop and Think

What do you notice about the angles in Example D?

Exercise 3

Pick any points A and B on the circumference of the circle, and draw an inscribed angle and a central angle from them on the same circle. Do this three times, with different points A and B. Write down the measure of your angles. What do you notice?

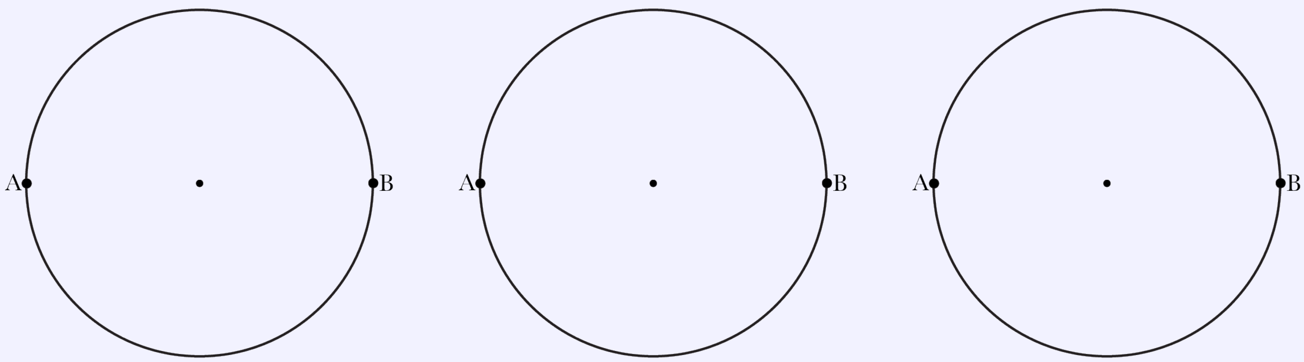


Exercise 3 Solution

Solutions may vary. You should notice that for all three of your angle sets, the central angle is exactly double your inscribed angle.

Exercise 4

Draw three inscribed angles from the diameter (all with a different vertex). Write down the measure of your angles. What do you notice?



Exercise 4 Solution

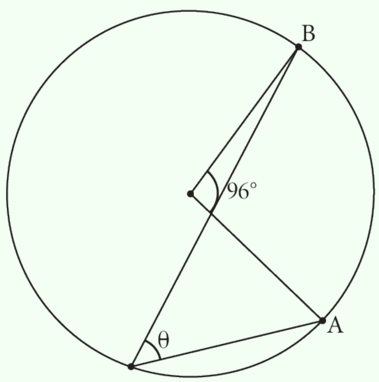
Solutions may vary. You should notice that all three inscribed angles from the diameter have a measure of exactly 90°.

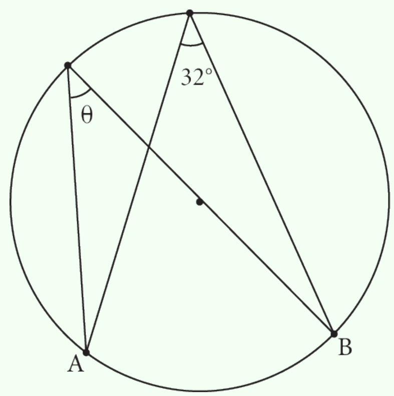
Given our observations, we can state a few true facts about the relationships between the angles inside of a circle:

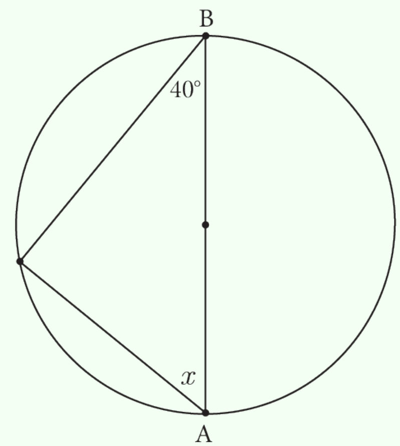
1. With respect to the same points A and B, the central angle is always twice the inscribed angle.
2. An inscribed angle from the diameter is always 90°.
3. All inscribed angles from the same points A and B will always be equal.

Example E

Solve for the indicated angles without the use of a protractor:







$\theta = 48°$ $\theta = 32°$ $x = 50°$

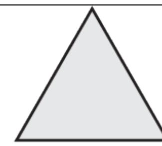
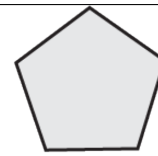
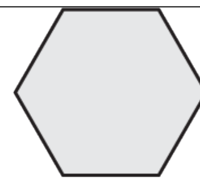
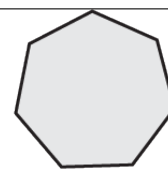
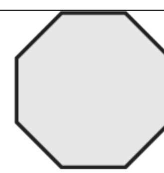
Angles All Around

Stop and Think

Is there anything you know about shapes and angles?

Regular Polygons are closed shapes with straight edges and vertices whose side lengths are all equal and whose angles are all equal. The **interior angles** of a polygon are always less than 180° and are on the *inside* of the shape. The **exterior angles** of a polygon are always *reflex angles* and are on the *outside* of the shape.

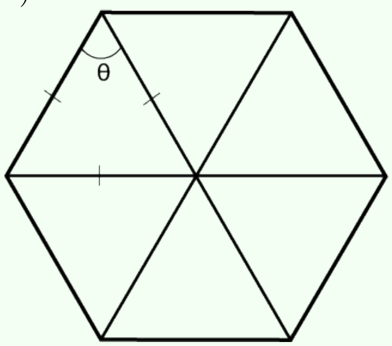
Let's fill in what we know about regular polygons in the table below. Although many polygons have proper names, we can generally refer to a polygon with n sides as an n-gon.

Number of sides	Interior Angle	Sum of Interior Angles	Image
3	$60°$	$180°$	 Equilateral Triangle
4	$90°$	$360°$	 Square
5	$108°$	$540°$	 Regular Pentagon
6	$120°$	$720°$	 Regular Hexagon
7	$128\frac{4}{7}°$	$900°$	 Regular Heptagon
8	$135°$	$1080°$	 Regular Octagon
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{180(n-2)}{n}$	$180(n-2)$	

Example F

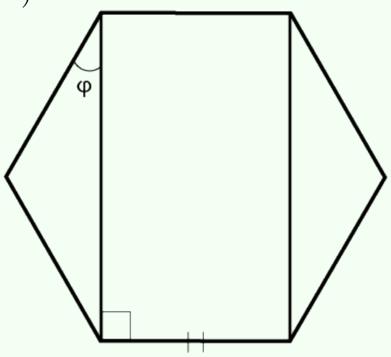
What's really cool about geometry is that you can combine different shapes! Let's see if, without any extra information, we can determine the measure of the missing angles in the diagrams below. Note that we are dealing with *regular* polygons.

a)



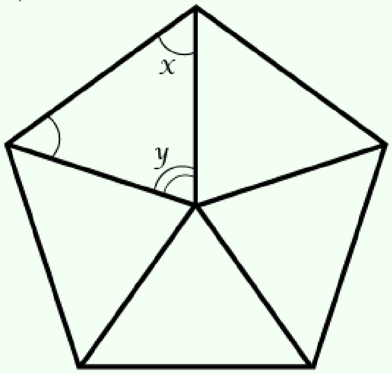
$$\theta = 60^\circ$$

b)



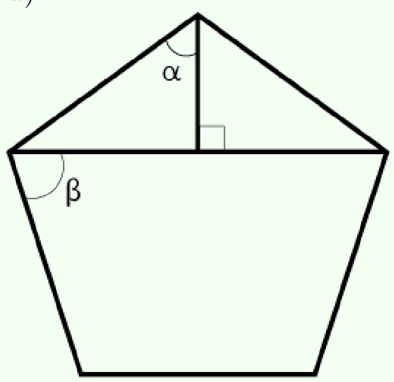
$$\begin{aligned}\varphi &= 120^\circ - 90^\circ \\ &= 30^\circ\end{aligned}$$

c)



$$\begin{aligned}x &= \frac{108^\circ}{2} \\ &= 54^\circ \\ y &= 180^\circ - 108^\circ \\ &= 72^\circ\end{aligned}$$

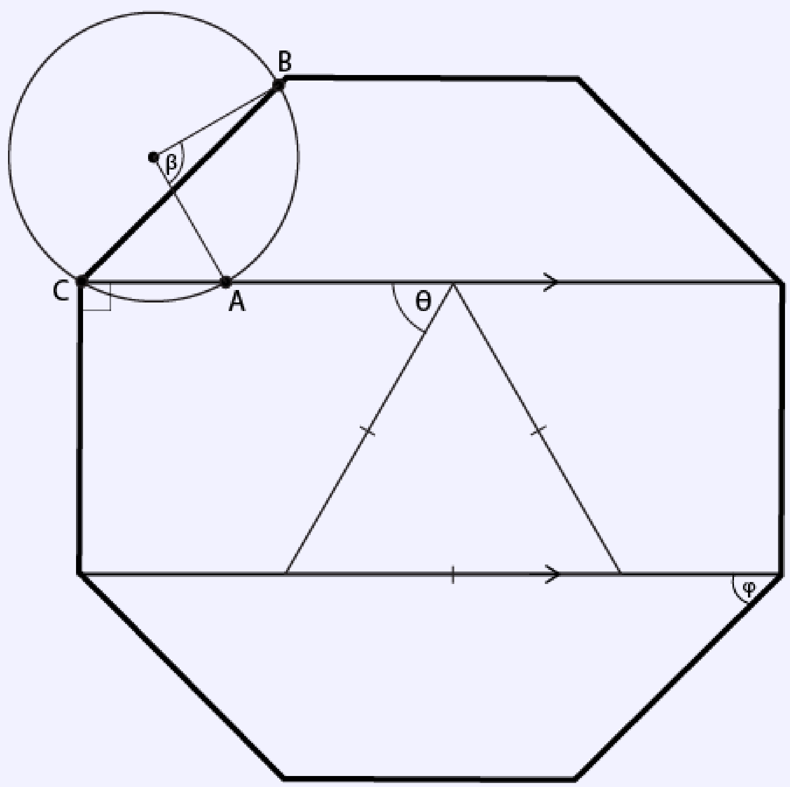
d)



$$\begin{aligned}\alpha &= \frac{108^\circ}{2}\\ &= 54^\circ\\ \beta &= 108^\circ - (180^\circ - 54^\circ - 90^\circ)\\ &= 108^\circ - 36^\circ\\ &= 72^\circ\end{aligned}$$

Exercise 5

Determine the measure of **all** the indicated angles (θ, β, φ) inside the regular octagon given below:

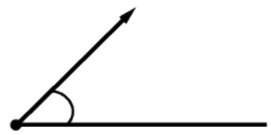
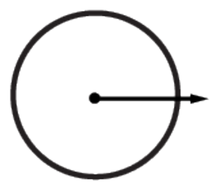
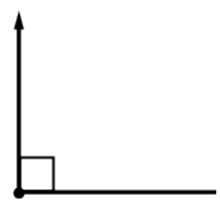
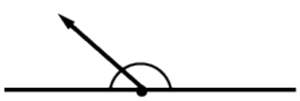
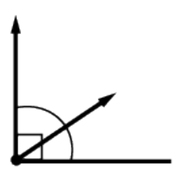
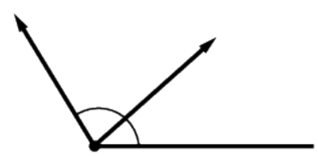
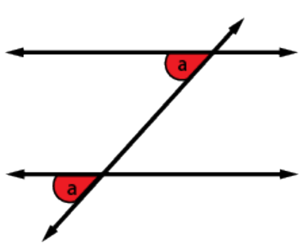
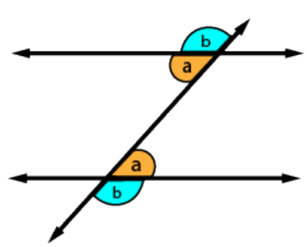
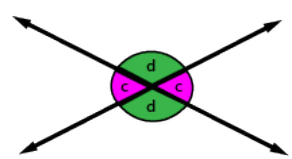


Exercise 5 Solution

In the middle of the octagon, there is an equilateral triangle, whose interior angles are all $60°$. Using the "Z-pattern", we can say that $\theta = 60°$.

Each interior angle inside a regular octagon is $135°$. Further, note that φ is adjacent to a right angle. Hence: $\varphi = 135° - 90° = 45°$.

Finally, notice that β is a central angle with a related inscribed angle at point C. This inscribed angle is $45°$ using the same reasoning as for φ, and thus $\beta = 2 \times 45° = 90°$.

 straight angle	 acute angle	 obtuse angle
 reflex angle	 complete angle (full rotation)	 right angle
 supplementary angles	 complementary angles	 adjacent angles
 corresponding angles	 alternate angles	 opposite angles